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Transient Flow in the Hydrodynamic Entrance Regions of Long Closed Ducts

CARL G. DOWNING

Oregon State University, Corvallis, Oregon

Velocity profiles were calculated for transient fluid flow in the hydrodynamic entrance regions of a long narrow slit and of a long tube. The solutions were obtained by use of a linearized momentum equation. An estimate was also made of the manner in which the resistance to flow of the entrance region develops with time.

The theoretical approach used by Sparrow, Lin, and Lundgren (2) in their analysis of steady flow in the entrance regions of closed ducts has been extended to the corresponding transient flow situation.

VELOCITY PROFILES

Consider the unsteady laminar incompressible flow of a Newtonian fluid in the entrance regions of (A) a tube and (B) a slit formed between two large parallel plates. The flow is described by the dimensional equations

$$\frac{\partial}{\partial x}(r^s u) + \frac{\partial}{\partial r}(r^s v) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left\{\frac{\partial^2 u}{\partial x^2} + \frac{1}{r^s}\frac{\partial}{\partial r}\left(r^s\frac{\partial u}{\partial r}\right)\right\} \tag{2}$$

where $s = 0$ and 1 for flow in a slit and in a tube, respectively. Taking the axial molecular transport of momentum to be negligible relative to the radial transport, and assuming that the pressure is constant across the section, one can linearize Equation (2):

$$\frac{\partial u}{\partial t} + \epsilon(x)\, U_\infty \frac{\partial u}{\partial x} = \Lambda(x, t) + \nu\frac{1}{r^s}\frac{\partial}{\partial r}\left(r^s\frac{\partial u}{\partial r}\right) \tag{3}$$

where U_∞ is the average steady velocity and $\Lambda(x, t)$ includes the pressure gradient and residual inertia terms. The $\epsilon(x)$ are the same stretching parameters defined and evaluated by reference 2 and discussed by reference 3; they will not be further discussed here other than to note that $\epsilon(x)$ for a tube ranges from a value of 0.42 at $x = 0$ to an asymptotic value of 1.82 at large x, and that corresponding values for a slit are 0.37 and 1.135. Defining the following dimensionless variables

$$\omega = \frac{u}{U_\infty},\quad \eta = \frac{r}{R},\quad \tau = \frac{\nu t}{R^2},\quad \text{and}\quad X^* = \frac{\nu x^*}{R^2 U_\infty}$$

where $dx = \epsilon(x)\, dx^*$, Equation (2) becomes

$$\frac{\partial \omega}{\partial \tau} + \frac{\partial \omega}{\partial X^*} = \Lambda^*(X^*, \tau) + \frac{1}{\eta^s}\frac{\partial}{\partial \eta}\left(\eta^s\frac{\partial \omega}{\partial \eta}\right) \tag{4}$$

Integration of Equation (4) across the section yields

$$\left(\frac{1}{2^s}\right)\Lambda^*(X^*, \tau) = \frac{d}{d\tau}\int_0^1 \omega\, \eta^s\, d\eta - \left.\frac{\partial \omega}{\partial \eta}\right|_{\eta=1} \tag{5}$$

which shows that $\Lambda^*(X^*, \tau)$ may be written as the sum of a macroscopic inertia term and a viscous friction term. Substitution of Equation (5) into Equation (4) then yields the following formulation of the problem:

$$\frac{\partial \omega}{\partial \tau} + \frac{\partial \omega}{\partial X^*} = (2^s)\frac{d}{d\tau}\int_0^1 \omega\, \eta^s\, d\eta$$

Carl G. Downing is now a consulting engineer in Corvallis, Oregon.

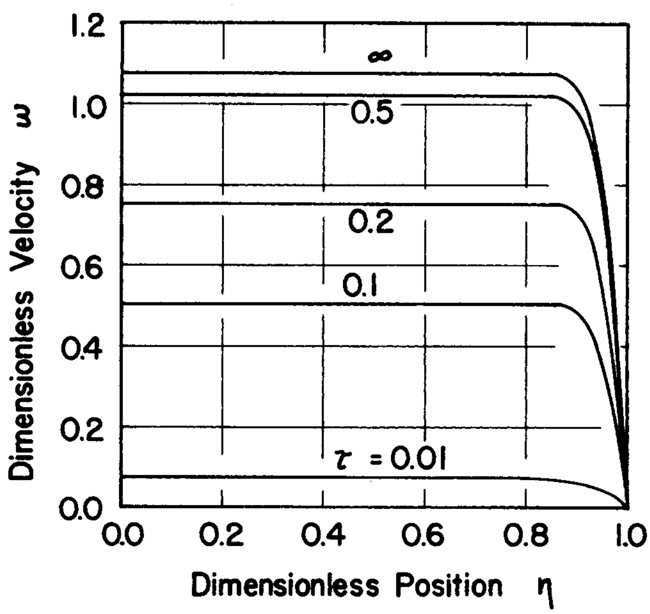

Fig. 1a. Velocity profiles for a tube; $X^* = 0.001$.

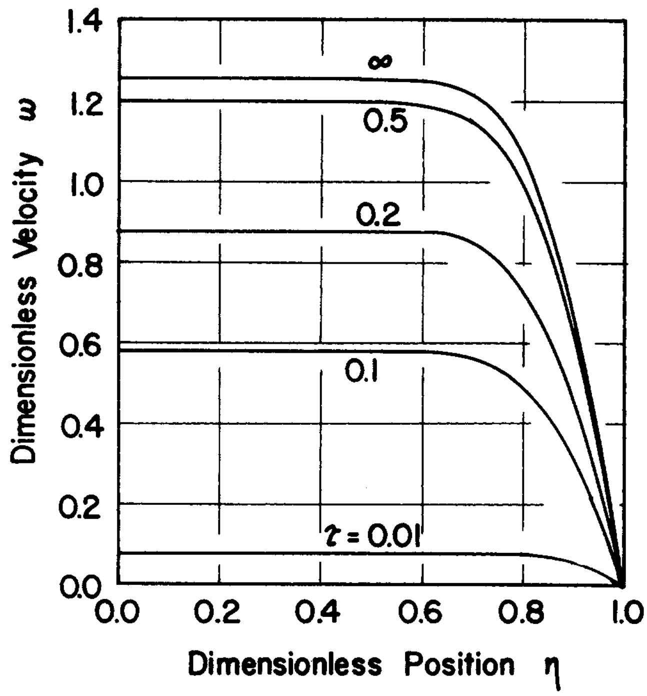

Fig. 1b. Velocity profiles for a tube; $X^* = 0.01$.

$$- (2^s) \left. \frac{\partial \omega}{\partial \eta} \right|_{\eta=1} + \frac{1}{\eta^s} \frac{\partial}{\partial \eta} \left(\eta^s \frac{\partial \omega}{\partial \eta} \right) \quad (6)$$

I.C. $\omega(X^*, \eta, 0) = 0$

B.C. 1 $\omega(0, \eta, \tau)$ independent of η

B.C. 2 $\frac{\partial \omega}{\partial \eta}(X^*, 0, \tau) = 0$

B.C. 3 $\omega(X^*, 1, \tau) = 0$

One additional specification must be made before Equation (6) can be solved. It will be assumed that the duct is long, so that the volumetric flow rate develops in the same manner as it would in an infinitely long tube subjected to a step change in pressure (compare reference *1*). With this additional restriction, Equation (6) can be solved using iterated Laplace transforms. The results are

A. *Tube flow*:

$$\omega = 2(1 - \eta^2) + \sum_{n=1}^{\infty} \left\{ \frac{4\,\phi_1 \exp(-\beta_n^2 \tau)}{\beta_n^2} \right\} \left\{ 1 - \begin{cases} 0, & 0 < \tau < X^* \\ \{1 - \exp(+\beta_n^2(\tau - X^*))\}, & \tau > X^* \end{cases} \right\}$$
$$- \sum_{l=1}^{\infty} \left\{ \frac{16\, J_0(\alpha_l \eta)}{\alpha_l^3 J_1(\alpha_l)} \right\} \exp(-\alpha_l^2 \tau)$$
$$+ \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \left\{ \frac{128\,\phi_1 \exp(-\beta_n^2 \tau)}{\alpha_l^4 \{\alpha_l^2 - \beta_n^2\}} \right\} \left\{ 1 - \begin{cases} 0, & 0 < \tau < X^* \\ \{1 - \exp(-\{\alpha_l^2 - \beta_n^2\}\{\tau - X^*\})\}, & \tau > X^* \end{cases} \right\} \quad (7a)$$

where

$$\phi_1 = \left\{ \frac{J_0(\beta_n \eta)}{J_0(\beta_n)} - 1 \right\}, \quad J_0(\alpha_l) = 0 \quad \text{and} \quad \frac{J_1(\beta_n)}{J_0(\beta_n)} = \frac{1}{2} \beta_n$$

B. *Slit flow*:

$$\omega = \frac{3}{2}(1 - \eta^2) + \sum_{n=1}^{\infty} \left\{ \frac{2\,\phi_0 \exp(-\beta_n^2 \tau)}{\beta_n^2} \right\}$$

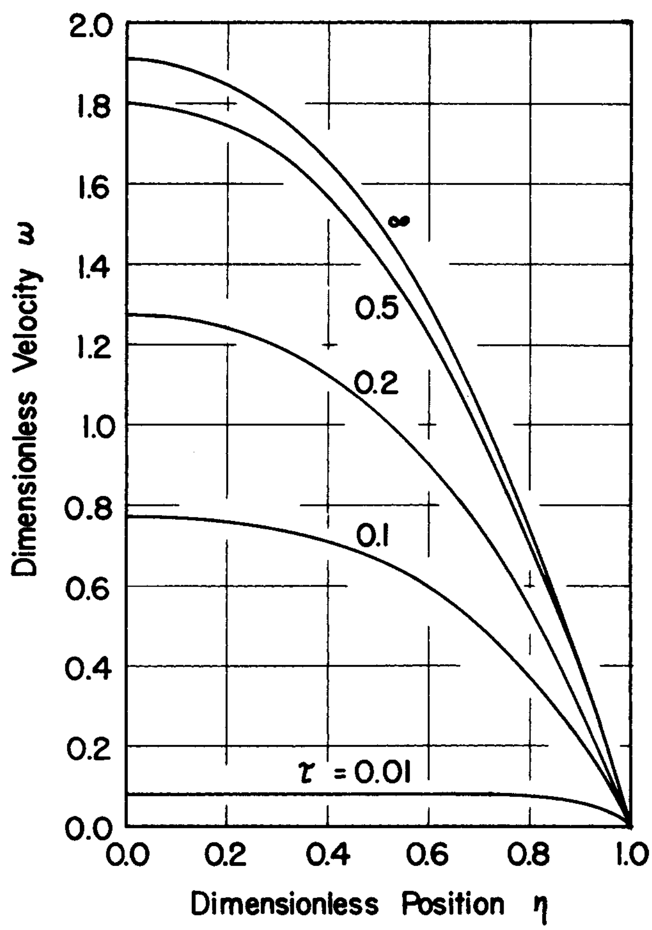

Fig. 1c. Velocity profiles for a tube; $X^* = 0.1$.

$$\left\{1-\begin{Bmatrix} 0, & 0<\tau<X^* \\ \{1-\exp\left(+\beta_n^2(\tau-X^*)\right)\}, & \tau>X^* \end{Bmatrix}\right\}$$

$$-\sum_{l=1}^{\infty}\left\{\frac{6(-1)^{l+1}\cos(\alpha_l\eta)}{\alpha_l^3}\right\}\exp(-\alpha_l^2\tau)$$

$$+\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left\{\frac{12\,\phi_0\exp(-\beta_n^2\tau)}{\alpha_l^4\{\alpha_l^2-\beta_n^2\}}\right\}$$

$$\left\{1-\begin{Bmatrix} 0, & 0<\tau<X^* \\ \{1-\exp\left(-\{\alpha_l^2-\beta_n^2\}\{\tau-X^*\}\right)\}, & \tau>X^* \end{Bmatrix}\right\} \tag{7b}$$

where

$$\left\{\frac{\cos(\beta_n\eta)}{\cos(\beta_n)}-1\right\}={}^0\phi\ ,\quad \alpha_l=\left(l-\frac{1}{2}\right)\pi,$$

$$\text{and}\quad \tan\beta_n=\beta_n\,.$$

These equations reduce to those given in reference *2* in the limit as $\tau \to \infty$.

Representative velocity profiles are plotted in Figures 1 and 2.

PRESSURE DROP

Integration of Equation (2) across the section and rearrangement yield an expression for dP/dx. Then separation of variables and integration from $P = P_0$ at $x = 0$ to $P = P$ at $x = x$ yield, in terms of dimensionless variables

$$\frac{1}{2(2^s)}\left\{\frac{P_0-P}{\frac{1}{2}\rho U_\infty^2}\right\}=\left[\int_0^1\omega^2\eta^s\,d\eta\right]_{X^*=0}^{X^*=X^*}$$

$$-\int_0^{X^*}\epsilon(X^*)\left.\frac{\partial\omega}{\partial\eta}\right|_{\eta=1}dX^*+\left\{\frac{d}{d\tau}\int_0^1\omega\,\eta^s\,d\eta\right\}X^0 \tag{8}$$

$$\text{where}\quad X^0=\int_0^{X^*}\epsilon(X^*)\,dX^*$$

If the resistance to flow due to the entrance is neglected, then the pressure gradient in the duct is constant, in keeping with the assumptions made in formulating the problem:

$$\left\{\frac{P_0-P}{\frac{1}{2}\rho U_\infty^2}\right\}_{\text{neglecting entrance effects}}=\left\{\begin{matrix}16\,X^0\text{ for tube flow}\\ 6\,X^0\text{ for slit flow}\end{matrix}\right\} \tag{9}$$

The pressure drop due to the entrance effects is thus the

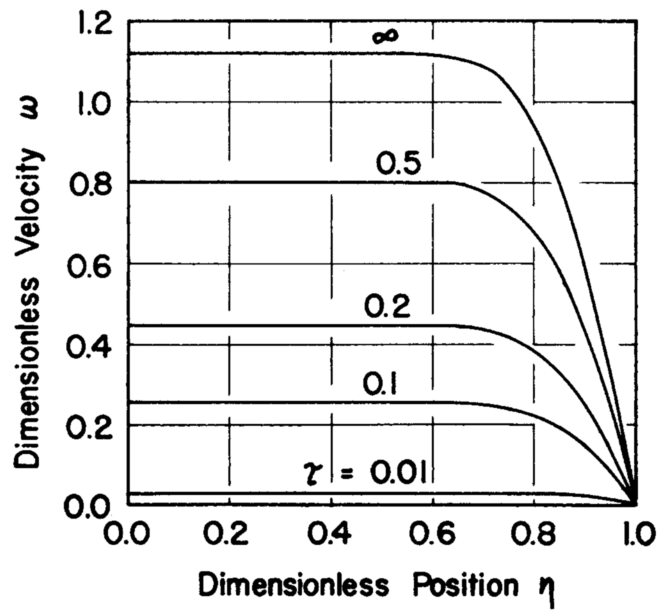

Fig. 2b. Velocity profiles for a slit; $X^* = 0.01$.

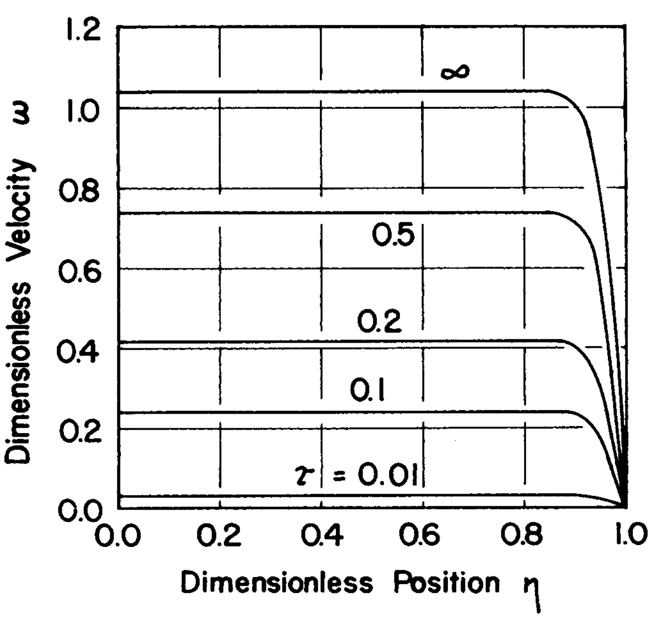

Fig. 2a. Velocity profiles for a slit; $X^* = 0.001$.

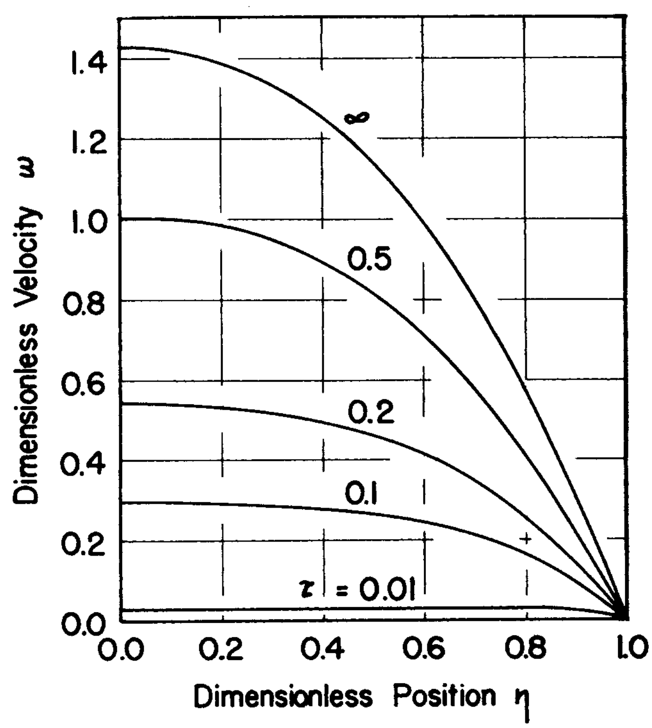

Fig. 2c. Velocity profiles for a slit; $X^* = 0.1$.

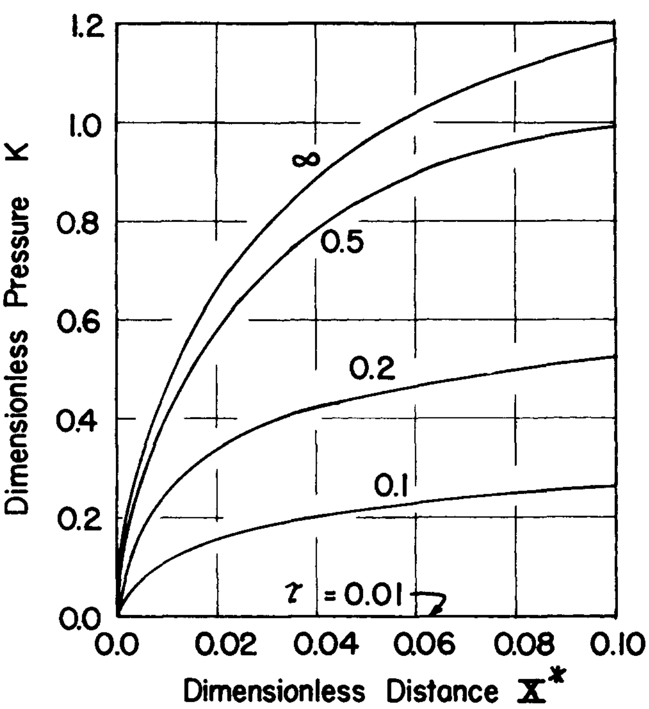

Fig. 3. Contribution of the entry section to the pressure drop in a tube.

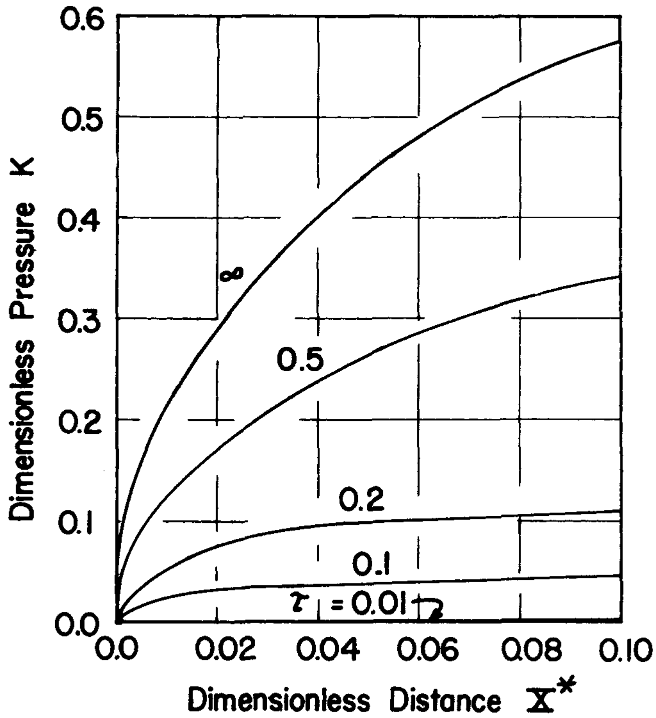

Fig. 4. Contribution of the entry section to the pressure drop in a slit.

difference between the two pressure drops calculated from Equations (8) and (9):

$$K = \left\{ \frac{P_0 - P}{\frac{1}{2}\rho U_\infty^2} \right\} - \left\{ \frac{P_0 - P}{\frac{1}{2}\rho U_\infty^2} \right\}_{\text{neglecting entrance effects}} \qquad (10)$$

The parameter K has been evaluated graphically and the results (estimated to be within $\pm$ 10%) are shown in Figures 3 and 4. For small τ, K increases slowly, since the velocity profiles are largely independent of X^*, just as they are in the reference duct in which there are no entrance effects. As the entrance region boundary layer develops, K increases to an asymptotic value for each X^*. The curves for K at $\tau \to \infty$ are the same as the corresponding curves given in reference 2.

DISCUSSION

Sparrow et al. (2), in their analysis of the corresponding steady flow case, used as their starting point the following linearized momentum equation:

$$\epsilon(x)\, U_\infty \frac{\partial u}{\partial x} = \Lambda(x) + \nu \frac{1}{r^s} \frac{\partial}{\partial r}\left(r^s \frac{\partial u}{\partial r} \right) \qquad (11)$$

The assumptions entering into the development of Equation (11) are that the axial molecular transport of momentum is negligible relative to the radial transport, and the pressure is constant across each section. Both assumptions break down at small distances from the duct inlet, but Sparrow et al. nevertheless found that for flow in tubes their velocity predictions were "... in remarkably good agreement with [selected] experimental data ... over the entire length of the entrance region," and that their corresponding pressure-drop predictions were in "very good" agreement with experimental data. No such data are available for the transient flow case. Comparison of Equations (3) and (11) suggests, however, that the calculated values presented in the present paper should certainly be valid at moderately long times. They should also be of reasonable accuracy at short times, although verification of this statement will require comparison with experimental data.

NOTATION

K = defined by Equation (10)
P; P_0 = pressure; pressure at the entrance cross section
r = radial coordinate for tube; distance from center line for slit
R = radius of tube or half-thickness of slit
s = 1 for tube; 0 for slit
t = time
u, v = velocity components in the x and r directions, respectively
U_∞ = average velocity as $\tau \to \infty$
x = axial coordinate, measured from duct entrance
x^* = stretched axial coordinate
X^0; X^* = $\nu x/R^2U_\infty$; $\nu x^*/R^2U_\infty$

Greek Letters

α, β = eigenvalues as defined by Equations (7)
ϵ = dimensionless stretching parameter
η = r/R
Λ = defined by Equation (5)
Λ^* = $R^2\Lambda/\nu U_\infty$
ν = kinematic viscosity
ρ = density
τ = $\nu t/R^2$
ϕ = defined by Equations (7)
ω = dimensionless velocity in x direction

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